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Finite-Dimensional Quantum Mechanics of a Particle. III. The Weylian Quantum Mechanics of Confined Quarks

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A finite-dimensional analog of Weyl's formulation of quantum kinematics of a physical system through irreducible Abelian groups of unitary ray rotations in system space offers many possibilities for the quantum mechanics of confined particles. This paper is devoted to the expansion of the recently developed framework of such Weylian finite-dimensional quantum mechanics which may provide a new way of thinking about the characteristics of quark physics.

1. INTRODUCTION

The theory of finite-dimensional quantum mechanics developed in recent contributions to this journal by the author and collaborators (Jagannathan et al., 1981; Jagannathan and Santhanam, 1982, hereafter referred to as I and II) suggests that for confined particles finite-dimensional matrices consistent with the Weyl relation

$$\exp(i\sigma\hat{p}_q/\hbar)\exp(i\tau\hat{q}/\hbar) = \exp(i\sigma\tau/\hbar)\exp(i\tau\hat{q}/\hbar)\exp(i\sigma\hat{p}_q/\hbar) \quad (1)$$

corresponding to the Heisenberg rule

$$[\hat{q}, \hat{p}_q] = i\hbar \quad (2)$$

should replace the customary infinite-dimensional Schrödinger representations for position and momentum operators. In this paper it is noted that the framework of the finite-dimensional quantum mechanics presented in I

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and II admits further expansion with interesting consequences for the physics of quarks.

At the basis of the development of our new theory of quantum mechanics of confined particles are the following: the fundamental works of Weyl (1932) and Schwinger (1960a, 1960b) on the understanding of the Heisenberg–Schrödinger formulation of quantum kinematics through the steps of finite-dimensional realizations of (1); the contributions of Alladi Ramakrishnan and collaborators to the mathematical aspects and physical applications of generalized Clifford algebras with generating relations of the type (1) (cf. Alladi Ramakrishnan, 1971, 1972; Alladi Ramakrishnan and Jagannathan, 1976; Jagannathan and Ranganathan, 1974, 1975; Jagannathan, 1978, and references therein for details on generalized Clifford algebras); and certain ideas on a form of quantum mechanics in discrete space based on (1) pursued by Santhanam (cf. Santhanam and Tekumalla, 1976; Santhanam, 1977a, 1977b, 1978). Other sources of encouragement for investigations in this direction are the currency of space-time lattice approaches, though purely on a technical basis, to the gauge theories of quarks since the advent of Wilson's model in 1974 (cf., e.g., Kadanoff, 1977; Kogut, 1979, for reviews and detailed references) and the occurrence of commutation relations of the type (1) in the discussions of quark confinement (cf. 't Hooft, 1981).

As already emphasized in I and II in our theory the configuration space of a particle is considered to have an eigenstructure derived from certain dynamical characteristics of the particle and as usual time is regarded to be an independent continuous parameter. In other words according to our theory position of a particle is an observable with a quantized spectrum just like energy, angular momentum, spin, etc., whereas the physical space itself is thought of only as a continuous manifold. The philosophy underlying this theory is that the physics of a particle with natural permanent confinement may involve only a position operator that has all its eigenvalues realizable within the region of confinement and hence in such a case the use of the customary Schrödinger operator with an unbounded spectrum, though along with suitable boundary conditions on the wave function, may be improper.

The idea of space-time quantization or the existence of some fundamental length and time has a long history in modern science starting with conceptual discussions like by Riemann in 1854 and by Einstein in 1921 and the initial applications to mathematical models of physical interest by Watagin, Markov, Ambartsumyan and Ivanenko, Heisenberg, Snyder, Yang, and others (cf., e.g., Blokhinstev, 1973; Dober and Yankovsky, 1968; Heisenberg, 1966; Ginsburg, 1976; Vyaltsev, 1965, for surveys and detailed references). Particularly in connection with the theories of elementary particles the nature of the microstructure of space-time is being repeatedly discussed (cf., e.g., Barut and Bracken, 1980; Cole, 1973; Dadic and Pisk, 1979; Ehrlich, 1978; Finkelstein, Frye and Susskind, 1974; Kadyshevsky, 1978; Lorente, 1981; Recami, 1981; Saavedra and Utreras, 1981; Stovicek and Tolar, 1979; Tati, 1980; Zidell, 1981, for recent discussions of the subject and also references to the earlier literature). Hence to distinguish the finite-dimensional quantum theory developed by us inspired by the statement of Weyl (1932) on the significance of the finite-dimensional realizations of (1) with regard to the fundamental laws of Nature at the level of nuclear physics from other forms of quantum mechanics based on different versions of space-time quantization[1] our theory derived in I, II, and the present paper may be called "Weylian finite-dimensional quantum mechanics" (WFDQM) or "Weylian quantum mechanics" (WQM) in short.

In Section 2 a précis of the formalism of the WFDQM is presented generalizing the basic structure derived in I and II by way of inclusion of a few more parameters consistent with the fundamental principles involved. In Section 3 the forms of nonrelativistic and relativistic wave equations in the WFDQM are discussed briefly. In conclusion the permanent quark confinement phenomenon is viewed in the light of the WFDQM in Section 4.

2. A PRÉCIS OF THE WFDQM

For any confined particle the governing quantum mechanics, the WFDQM, is supposed to be determined by a set of "space quantum numbers," J, M, S, and λ, associated with the particle such that

$$J, M \in \{1,2,3,\ldots\}, \qquad S \in \{0,1,2,\ldots\}, \qquad 0<\lambda<\infty \tag{3}$$

$$M < 2J+1, \qquad \text{g.c.d.}\,(M, 2J+1) = 1 \tag{4}$$

Any given set of space quantum numbers (J, M, S, λ) defines for the corresponding particle a "quantum of position" ε and a "quantum of momentum" η through

$$\varepsilon = \lambda\left(2\pi\hbar^2 G\{S+[M/(2J+1)]\}/e^2c^2\right)^{1/2} \tag{5}$$

$$\eta = \lambda^{-1}\left(2\pi e^2c^2\{S+[M/(2J+1)]\}/G\right)^{1/2} \tag{6}$$

[1]Recently Gudder and Naroditsky (1981, *International Journal of Theoretical Physics*, **20**, 619) also have given a formalism of finite-dimensional quantum mechanics very similar to ours in structure but with somewhat basically different assumptions on the nature of the spectra of position and momentum.

with

$$2\pi\{S+[M/(2J+1)]\}=\xi=\varepsilon\eta/\hbar \tag{7}$$

$$\lambda=\left(e^2c^2\varepsilon/\hbar G\eta\right)^{1/2} \tag{8}$$

representing the two simple dimensionless expressions involving ε, η, and the universal constants, e, c, $\hbar$, and G. Then the fundamental postulates of the WFDQM of the particle are as follows: (I) The radial coordinate r of the particle defined with respect to the center of a sphere of confinement has only the eigenvalues

$$r_n=n\varepsilon,\qquad n=1,2,\ldots,2J+1 \tag{9}$$

Consequently the corresponding radial coordinate operator R in the position representation is the $(2J+1)$-dimensional matrix with elements

$$\langle n|R|n'\rangle=\varepsilon\langle n|N|n'\rangle=\varepsilon n\delta_{nn'}$$

$$n,n'=1,2,\ldots,2J+1 \tag{10}$$

(II) The radial momentum of the particle p_r has the eigenvalues

$$p_{rn}=n\eta,\qquad n=0,\pm 1,\pm 2,\ldots,\pm J \tag{11}$$

The corresponding radial momentum operator P_r is defined to be conjugate to the radial coordinate operator R such that

$$\exp(i\varepsilon P_r/\hbar)\exp(i\eta R/\hbar)=\exp(i\varepsilon\eta/\hbar)\exp(i\eta R/\hbar)\exp(i\varepsilon P_r/\hbar) \tag{12}$$

like in the Weyl relation (1). Then P_r is given in the position representation by the $(2J+1)$-dimensional matrix with elements

$$\langle n|P_r|n'\rangle=\begin{cases}0 & \text{if } n=n' \\ (in'\eta/2n)\operatorname{cosec}[2\pi JM(n-n')/(2J+1)] & \text{if } n\neq n'\end{cases}$$

$$n,n'=1,2,3,\ldots,2J+1 \tag{13}$$

(III) The angular coordinates θ and φ in any spherical polar coordinate system defined with the center of the sphere of confinement as the origin have finite and continuous spectra as usual with the values

$$0 \leqslant \theta < \pi, \qquad 0 \leqslant \varphi < 2\pi \tag{14}$$

Thus in the (r, θ, φ) representation the state vector of the particle, $|\Psi(t)\rangle$, has $(2J+1)$ components with each component as a function of θ and φ and the operator corresponding to any observable is to be derived from its normal quantum counterpart in the (r, θ, φ) representation simply by the replacement rule

$$\hat{r} \rightarrow R, \qquad \hat{p}_r \rightarrow P_r \tag{15}$$

without any change with regard to θ and φ. Except for this replacement of the customary operators $\hat{r}$ and $\hat{p}_r$ for radial coordinate and momentum, respectively, by the relevant finite-dimensional analogs R and P_r and the consequent changes in the mathematical operations all other formal aspects of the usual Heisenberg–Schrödinger quantum theory are to be adopted in the obviously straightforward manner whether it is nonrelativistic or relativistic case, i.e., whether the finite-dimensional Hamiltonian is derived using the rule (15) from nonrelativistic or relativistic normal quantum Hamiltonian.

The following remarks are to be noted in connection with the above formalism.

(i) The development of the operator P_r is as follows.

$$P_r = R^{-1}PR = N^{-1}PN = \eta N^{-1}\Phi N \tag{16}$$

$$\langle n|N|n'\rangle = n\,\delta_{nn'}, \qquad n, n' = 1,2,3,\ldots,2J+1 \tag{17}$$

$$\langle n|\Phi|n'\rangle = \begin{cases} 0 & \text{if } n = n' \\ (i/2)\operatorname{cosec}[2\pi JM(n-n')/(2J+1)] & \text{if } n \neq n' \end{cases}$$

$$n, n' = 1,2,3,\ldots,2J+1 \tag{18}$$

$$\Phi = \mathbb{S}\mathfrak{N}\mathbb{S}^{-1} = \mathbb{S}\mathfrak{N}\mathbb{S}^{+} \tag{19}$$

$$\langle s|\mathfrak{N}|s'\rangle = s\,\delta_{ss'},\ s, s' = -J, -J+1,\ldots,-1,0,1,\ldots,J-1, J \tag{20}$$

$$\langle n|\mathbb{S}|s\rangle = (2J+1)^{-1/2}\omega^{ns}$$

$$n = 1,2,\ldots,2J+1,\ s = -J, -J+1,\ldots,-1,0,1,\ldots,J-1, J \tag{21}$$

$$\omega = \exp(i\xi) = \exp(i\varepsilon\eta/\hbar) = \exp[i2\pi M/(2J+1)] \tag{22}$$

The matrices N and Φ are seen to be Hermitian.

(ii) The conditions

$$\exp[i(2J+1)\varepsilon P_r/\hbar] = I \tag{23}$$

$$\exp[i(2J+1)\eta R/\hbar] = I \tag{24}$$

with I as the $(2J+1)$-dimensional unit matrix, subsidiary to (12), help fixing the spectra of r and p_r uniquely as in (9) and (11). The relations (12), (23), and (24) are easily verified by observation that if

$$\mathfrak{A} = \exp(i\varepsilon P_r/\hbar) = N^{-1}\exp(i\varepsilon P/\hbar)N \tag{25}$$

$$\mathfrak{B} = \exp(i\eta R/\hbar) \tag{26}$$

then

$$\mathfrak{A} = N^{-1}AN = \begin{pmatrix} 0 & 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{3}{2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{4}{3} & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & (2J+1)/2J \\ 1/(2J+1) & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{27}$$

$$\mathfrak{B} = \begin{pmatrix} \omega & & & & & \\ & \omega^2 & & & \varnothing & \\ & & \omega^3 & & & \\ & & & \ddots & & \\ & \varnothing & & & \omega^{2J} & \\ & & & & & 1 \end{pmatrix} \tag{28}$$

satisfying

$$\mathfrak{A}\mathfrak{B} = \omega\mathfrak{B}\mathfrak{A} \tag{29}$$

$$\mathfrak{A}^{2J+1} = I \tag{30}$$

$$\mathfrak{B}^{2J+1} = I \tag{31}$$

in accordance with (12), (13) and (24), respectively. It is to be recalled that the unitary irreducible representation of the set of relations

$$AB = \omega BA, \qquad A^{2J+1} = I, \qquad B^{2J+1} = I$$

$$\omega = \exp[i2\pi M/(2J+1)], \qquad \text{g.c.d.}(M, 2J+1) = 1 \tag{32}$$

is provided by

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & & & & \varnothing \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ \varnothing & & & & \omega^{2J} \end{pmatrix} \tag{33}$$

uniquely up to equivalence (Weyl, 1932).

(iii) The choice of an even integer as the dimension of the matrices R and P_r and the natural requirement of a symmetry between the positive and negative parts of the spectrum of P_r lead to the exclusion of the zero eigenvalue for p_r implying an *a priori* unreasonable assumption of a sort of inherent eternal motion for the particle. Hence the dimension of the matrices R and P_r has been taken here to be an odd integer. If needed the even-dimensional case for R and P_r can be readily developed in a manner parallel to the odd-dimensional case.

(iv) The above description uses spherical polar coordinates defined in a frame stationary with respect to the center of the sphere of confinement of the particle fixed as the origin and in this preferred frame the eigenstructure of the configuration space of the particle has rotational invariance. The corresponding picture in any other frame can be obtained by the usual coordinate transformation formulas.

(v) In treating any system involving more than one particle the usual direct-product formalism must be employed.

3. WAVE EQUATIONS IN THE WFDQM

As examples of using the above formalism the forms taken by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\ |\Psi(t)\rangle=\hat{\mathcal{H}}\ |\Psi(t)\rangle \tag{34}$$

in the WFDQM will be considered below for (i) the nonrelativistic case of a particle under the influence of a central potential $V(r)$, (ii) the relativistic case of a free spin-0 particle of the Klein–Gordon type, and (iii) the relativistic case of a free spin-1/2 particle of the Dirac type when each of them belongs to a specific (J, M, S, λ) representation of the WFDQM. Here by "free" is meant that the forces acting on the particle inside the sphere of confinement are neglected and the inclusion of such forces can be done as is customary in the normal quantum theory, of course using the finite-dimensional equivalents of the relevant operators. Then the results in the three cases mentioned are as follows.

(i) For the nonrelativistic case of a particle of effective mass μ under the influence of a central potential $V(r)$, (34) takes the form

$$i\hbar\frac{\partial}{\partial t}\ |\Psi(t)\rangle=\left[-(\hbar^2/2\mu)\nabla^2+V(R)\right]\ |\Psi(t)\rangle \tag{35}$$

with

$$\nabla^2=-\hbar^{-2}(P_r{}^2+R^{-2}\hat{L}^2) \tag{36}$$

$$\hat{L}^2=-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] \tag{37}$$

For a stationary state with

$$|\Psi_{nlm}(t)\rangle=R^{-1}\ |\chi_{nlm}\rangle Y_{lm}(\theta,\varphi)\exp(-iE_{nlm}t/\hbar) \tag{38}$$

corresponding to a fixed set of angular momentum quantum numbers (l, m) the energy eigenvalues $\{E_{nlm}|n=1,2,\ldots,2J+1\}$ are to be determined from the radial matrix equation

$$\left\{(2\mu)^{-1}\left[P^2+l(l+1)\hbar^2R^{-2}\right]+V(R)\right\}\ |\chi_{nlm}\rangle=E_{nlm}\ |\chi_{nlm}\rangle \tag{39}$$

with $P=RP_rR^{-1}$ as defined in (16). The allowed values of l and m are given as usual by

$$l = 0, 1, 2, \ldots, \qquad m = 0, \pm 1, \pm 2, \ldots, \pm l \tag{40}$$

Using the matrices N and Φ defined in (17) and (18), (39) can be written in a dimensionless form as

$$\left[\Phi^2 + l(l+1)\xi^{-2}N^{-2} + 2\mu\eta^{-2}V(R)\right] |\chi_{nlm}\rangle = 2\mu\eta^{-2}E_{nlm}\, |\chi_{nlm}\rangle \tag{41}$$

For example when the potential $V(r)$ is the spherical harmonic potential $\frac{1}{2}kr^2$, (41) will have the form

$$\left[\Phi^2 + l(l+1)\xi^{-2}N^{-2} + \mu\eta^{-2}kR^2\right] |\chi_{nlm}\rangle = 2\mu\eta^{-2}E_{nlm}\, |\chi_{nlm}\rangle \tag{42}$$

For the case of a free particle with $V(r) = 0$ and $J = 1$ the energy spectrum can be obtained by substituting in (41) with $V(R) = 0$:

$$\Phi = \frac{(-1)^M i}{\sqrt{3}} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \tag{43}$$

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{44}$$

$$\xi = 2\pi[S + (M/3)], \qquad \eta = \lambda^{-1}\{2\pi e^2 c^2 [S + (M/3)]/G\}^{1/2}$$

$$M = 1, 2, \qquad S = 0, 1, 2, \ldots \tag{45}$$

for any choice of λ. The result can be expressed as

$$E_{1lm}^{\text{FNR}} = (\eta^2/6\mu)\{2 + 49x_0 - 2z_0\cos[(\pi - \Omega_0)/3]\} \tag{46}$$

$$E_{2lm}^{\text{FNR}} = (\eta^2/6\mu)\{2 + 49x_0 - 2z_0\cos[(\pi + \Omega_0)/3]\} \tag{47}$$

$$E_{3lm}^{\text{FNR}} = (\eta^2/6\mu)\{2 + 49x_0 + 2z_0\cos(\Omega_0/3)\} \tag{48}$$

with

$$x_0 = l(l+1)/36\xi^2, \qquad z_0 = (889x_0^2 + 1)^{1/2}$$

$$\Omega_0 = \text{minimum of } \cos^{-1}[(24013x_0^3 - 1)/z_0^3] \tag{49}$$

for any choice of (M, S, λ). The superscript FNR indicates free nonrelativistic case.

(ii) For the relativistic case of a free spin-0 Klein–Gordon particle the equivalent of (34) is

$$-\hbar^2\frac{\partial^2}{\partial t^2}\ |\Psi(t)\rangle = \left(-\hbar^2c^2\nabla^2 + \mu^2c^4\right)\ |\Psi(t)\rangle \tag{50}$$

where ∇^2 is the same as defined through (36) and (37) and μ is the effective mass of the particle. Now for the stationary states

$$|\Psi_{nlm}(t)\rangle = R^{-1}\ |\chi_{nlm}\rangle Y_{lm}(\theta,\varphi)\exp(-iE_{nlm}t/\hbar)$$
$$n=1,2,\ldots,2J+1, \qquad l=0,1,2,\ldots, \qquad m=0,\pm1,\pm2,\ldots,\pm l \tag{51}$$

the energy eigenvalues $\{E_{nlm}\}$ are determined by

$$\left[c^2P^2 + \hbar^2c^2l(l+1)R^{-2} + \mu^2c^4\right]\ |\chi_{nlm}\rangle = E_{nlm}^2\ |\chi_{nlm}\rangle \tag{52}$$

or

$$\left[\Phi^2 + l(l+1)\xi^{-2}N^{-2}\right]\ |\chi_{nlm}\rangle = (\eta c)^{-2}\left(E_{nlm}^2 - \mu^2c^4\right)\ |\chi_{nlm}\rangle \tag{53}$$

Comparing (53) and (41) with $V(R)=0$ it is obvious that in the case of the free Klein–Gordon particle the energy spectrum is given by

$$E_{nlm}^{\text{FKG}} = \left(2\mu c^2E_{nlm}^{\text{FNR}} + \mu^2c^4\right)^{1/2} \tag{54}$$

where the solutions of (41) give $\{E_{nlm}^{\text{FNR}}\}$.

(iii) For the case of a free spin-1/2 Dirac particle of effective mass μ, (34) takes the form

$$i\hbar\frac{\partial}{\partial t}\ |\Psi(t)\rangle = \mathcal{H}_D\ |\Psi(t)\rangle$$
$$= \left[c\alpha_r\otimes P_r + i\hbar c\alpha_r\hat{K}\otimes R^{-1} + \mu c^2\beta\otimes I\right]\ |\Psi(t)\rangle \tag{55}$$

where the spinor $|\Psi(t)\rangle$ has $4(2J+1)$ components,

$$\alpha_r = \begin{pmatrix} 0 & 0 & \cos\theta & \sin\theta\exp(-i\varphi) \\ 0 & 0 & \sin\theta\exp(i\varphi) & -\cos\theta \\ \cos\theta & \sin\theta\exp(-i\varphi) & 0 & 0 \\ \sin\theta\exp(i\varphi) & -\cos\theta & 0 & 0 \end{pmatrix} \tag{56}$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{57}$$

$$\hat{K} = \hbar^{-2}(\hat{J}^2 - \hat{L}^2) + \tfrac{1}{4} \tag{58}$$

and $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$ represents the customary operator for the total angular momentum including spin. The stationary state solutions of (55) are given by

$$|\Psi_{njmp}(t)\rangle = \tfrac{1}{2}\exp(-iE_{njmp}t/\hbar) \times$$

$$\begin{pmatrix} \{(1-p)[(j+m)/2j]^{1/2} - (1+p)[(j-m+1)/(2j+2)]^{1/2}\} Y_{j+(1/2)p,\,m-1/2}(\theta,\varphi) R^{-1} |F_{njmp}\rangle \\ \{(1-p)[(j-m)/2j]^{1/2} + (1+p)[(j+m+1)/(2j+2)]^{1/2}\} Y_{j+(1/2)p,\,m+1/2}(\theta,\varphi) R^{-1} |F_{njmp}\rangle \\ i\{(1+p)[(j+m)/2j]^{1/2} - (1-p)[(j-m+1)/(2j+2)]^{1/2}\} Y_{j-(1/2)p,\,m-1/2}(\theta,\varphi) R^{-1} |G_{njmp}\rangle \\ i\{(1+p)[(j-m)/2j]^{1/2} + (1-p)[(j+m+1)/(2j+2)]^{1/2}\} Y_{j-(1/2)p,\,m+1/2}(\theta,\varphi) R^{-1} |G_{njmp}\rangle \end{pmatrix}$$

$$n = 1, 2, \ldots, 2J+1, \quad j = 1/2, 3/2, \ldots, \quad m = \pm 1/2, \pm 3/2, \ldots, \pm j, \quad p = \pm 1 \tag{59}$$

such that

$$\hat{J}^2 \; |\Psi_{njmp}(t)\rangle = j(j+1)\hbar^2 \; |\Psi_{njmp}(t)\rangle \tag{60}$$

$$\hat{J}_z \; |\Psi_{njmp}(t)\rangle = m\hbar \; |\Psi_{njmp}(t)\rangle \tag{61}$$

$$\hat{S}^2 \; |\Psi_{njmp}(t)\rangle = \tfrac{3}{4}\hbar^2 \; |\Psi_{njmp}(t)\rangle \tag{62}$$

$$\beta \mathcal{P} \; |\Psi_{njmp}(t)\rangle = (-1)^{j+(1/2)p} \; |\Psi_{njmp}(t)\rangle \tag{63}$$

$$\beta \hat{K} \; |\Psi_{njmp}(t)\rangle = -(j+\tfrac{1}{2})p \; |\Psi_{njmp}(t)\rangle \tag{64}$$

$$\mathcal{H}_D \; |\Psi_{njmp}(t)\rangle = E_{njmp} \; |\Psi_{njmp}(t)\rangle \tag{65}$$

Here p characterizes the parity of the states through (63) in which $\mathcal{P}$ represents the usual parity operation. (For details on the standard treatment of the Dirac equation in polar coordinates from which the above derivations follow cf., e.g., Alonso and Valk, 1973.) The eigenvalue equation (65) for energy simplifies to the set of coupled equations

$$\left[icP + \hbar c p (j+\tfrac{1}{2}) R^{-1}\right] \; |F_{njmp}\rangle = \left(E_{njmp} + \mu c^2\right) \; |G_{njmp}\rangle$$

$$\left[-icP + \hbar c p (j+\tfrac{1}{2}) R^{-1}\right] \; |G_{njmp}\rangle = \left(E_{njmp} - \mu c^2\right) \; |F_{njmp}\rangle \tag{66}$$

from which it follows that

$$\left\{\Phi^2 + \left(j+\tfrac{1}{2}\right)^2 \xi^{-2} N^{-2} + ip\left(j+\tfrac{1}{2}\right)\xi^{-1}[N^{-1}, \Phi]\right\} |F_{njmp}\rangle$$
$$= (\eta c)^{-2}\left(E_{njmp}^2 - \mu^2 c^4\right) |F_{njmp}\rangle \tag{67}$$

$$|G_{njmp}\rangle = \left[\eta c/\left(E_{njmp} + \mu c^2\right)\right]\left[i\Phi + p\left(j+\tfrac{1}{2}\right)\xi^{-1}N^{-1}\right] |F_{njmp}\rangle \tag{68}$$

When $J = 1$ the solution of (67) with the help of (43)–(45) leads to the energy spectrum of the free Dirac particle as given by

$$E_{1jmp}^{\mathrm{FD}} = \left(\mu^2 c^4 + (\eta^2 c^2/3)\{2 + 49x - 2z\cos[(\pi - \Omega)/3]\}\right)^{1/2} \tag{69}$$

$$E_{2jmp}^{\mathrm{FD}} = \left(\mu^2 c^4 + (\eta^2 c^2/3)\{2 + 49x - 2z\cos[(\pi + \Omega)/3]\}\right)^{1/2} \tag{70}$$

$$E_{3jmp}^{\mathrm{FD}} = \{\mu^2 c^4 + (\eta^2 c^2/3)[2 + 49x + 2z\cos(\Omega/3)]\}^{1/2} \tag{71}$$

with

$$x = \left(j+\tfrac{1}{2}\right)^2/36\xi^2, \qquad z = \left(889x^2 + 312y^2 + 1\right)^{1/2},$$
$$y = (-1)^M p\left(j+\tfrac{1}{2}\right)/12\sqrt{3}\,\xi$$
$$\Omega = \text{minimum of } \cos^{-1}\left[\left(24013x^3 + 11268xy^2 + 2592y^3 + 468y^2 - 256xy - 1\right)/z^3\right] \tag{72}$$

for all values of M and S from $\{M = 1, 2,\ S = 0, 1, 2, \ldots\}$ and any choice of λ.

4. CONCLUSION: QUARK CONFINEMENT AND THE WFDQM

The motivation for the development of the WFDQM is as follows. Just like the transition from classical to quantum mechanics in the domain of atomic physics it is likely that in the realm of subnuclear physics the quantum kinematical basis itself undergoes a transition from the Heisenberg–Schrödinger phase with infinite-dimensional representations of position and momentum operators to a finite-dimensional Weyl phase consistent with the expression (1) of Heisenberg's canonical commutation relation (2). The physical realization of this phase transition of quantum theory can happen in two ways as follows.

(i) For any particle the space quantum numbers are not constants but vary in such a way that when $J \to \infty$,

$$\langle n|R|n'\rangle \to \langle r|\hat{r}|r'\rangle = r\delta(r-r') \tag{73}$$

$$\langle n|P_r|n'\rangle \to \langle r|\hat{p}_r|r'\rangle = \left[\frac{-i\hbar}{r}\left(\frac{\partial}{\partial r}\right)r\right]\delta(r-r') \tag{74}$$

i.e., the Heisenberg–Schrödinger quantum theory is the asymptotic approximation of the WFDQM in accordance with the general philosophy of Bohr's correspondence principle. For instance, for a particle of mass μ confined to be within a sphere of radius ρ the model presented in I and II suggests, in the language of the present paper, that

$$J = \text{integer part of } (1/2)\left[(\rho^2\mu^2c^2/2\pi\hbar^2)-1\right] \tag{75}$$

$$M=1, \qquad S=0, \qquad \xi = 2\pi/(2J+1), \qquad \text{for any } J \tag{76}$$

$$\lambda = e/\mu G^{1/2}, \qquad \text{for any } J \tag{77}$$

$$\varepsilon = [2\pi/(2J+1)]^{1/2}\hbar/\mu c, \qquad 0 \leqslant (\rho/\varepsilon)-(2J+1) < 1 \tag{78}$$

$$\eta = [2\pi/(2J+1)]^{1/2}\mu c \tag{79}$$

$$\lim \rho \to \infty, \qquad J\to\infty, \qquad \varepsilon\to 0, \qquad \eta \to 0, \qquad (2J+1)\varepsilon \to \infty,$$
$$J\eta\to\infty, \qquad R\to\hat{r}, \qquad P_r \to \hat{p}_r \tag{80}$$

Then it is also seen that the energy spectrum of the nonrelativistic spherical harmonic oscillator with $J=1$ considered in II reduces in the zero frequency limit to the free particle energy spectrum of (46)–(49) corresponding to the choice of parameters as in (76)–(79) with $J=1$. From this point of view the confinement phenomenon itself is not the result of the WFDQM governing the particle but whatever may be the forces causing confinement in any particular situation the WFDQM with proper space quantum numbers should provide the correct picture of the dynamics of the confined particle since the customary Heisenberg–Schrödinger quantum theory should be valid strictly only for an unconfined particle with $\rho=\infty$. Of course in practice even in the case of atomic and nuclear phenomena happening essentially within some bounded region of space the value of J can be so large, as for example in the model based on (75)–(80), that the usual quantum theory is quite adequate. However, for the case of quarks under permanent confinement within extremely small region of space the WFDQM corresponding to low values of J must replace the normal quantum mechanics just like quantum mechanics replaces classical mechanics in the realm of atomic physics. In this connection it is to be noted that the choice, $M = 1$, $S = 0$, for any J, as in (76), is the only permissible choice from this point of view which requires naturally that for any particle the change of formalism with ρ must be in a definite fashion and as $\rho \to \infty$ (80) is realized uniquely.

(ii) A radically different alternative to the point of view (i), advocated in I and II, is to consider that each particle has associated with it a set of fixed characteristic values for the space quantum numbers J, M, S, and λ. Just like a particle of definite mass and spin is thought of as corresponding to a particular representation of the Poincaré group each one of the physically distinct realizations of the WFDQM labeled by different sets of values for (J, M, S, λ) may be associated with its own particle. Thus all freely occurring or isolatable particles may correspond to very large values of J so that

$$J \approx \infty, \qquad M = 1, \qquad S = 0, \qquad \lambda = \left(e^2c^2\varepsilon/\hbar G\eta\right)^{1/2}$$
$$\varepsilon \approx 0, \qquad \eta \approx 0, \qquad (2J+1)\varepsilon \approx \infty, \qquad J\eta \approx \infty, \qquad (2J+1)\varepsilon\eta = 2\pi\hbar \tag{81}$$

and the usual Heisenberg–Schrödinger quantum theory is satisfied very well by them under all circumstances. When such a particle is trapped inside a finite spherical region its position and momentum operators do not change their structure, but only the wave function is subject to the boundary conditions implying the absence of the particle outside the region of confinement, as is customary in the current quantum mechanical treatment. Among such particles with the same large value for J intrinsic differences can occur owing to different values for λ or ε/η. This follows from the fact that the necessary and sufficient conditions to be satisfied for the Heisenberg–Schrödinger quantum theory to hold well are that $\varepsilon \approx 0$, $\eta \approx 0$, $(2J+1)\varepsilon \approx \infty$, $J\eta \approx \infty$, and $(2J+1)\varepsilon\eta = 2\pi\hbar$, whereas ε/η can take any positive value. Now the interesting situation arises when a particle belongs to low values of both J and ε. Such a "quark" will have a finite permanent characteristic confining radius, namely, $(2J+1)\varepsilon$, the maximum eigenvalue of its radial coordinate, and it can occur only as a constituent of a conglomeration of other quarks so that the resulting composite "hadron" can provide a rest frame for the formation of the sphere of confinement of its "partons". Inside this sphere of confinement the constituent quarks can be quite "free", obeying equations of the type considered in the previous section with "gluons" mediating the quantum jumps between the different levels of the energy spectra. The principal quantum number n with $(2J+1)$

possible values for any given J may be just what has been identified presently as the "color" of the quark and the quantum numbers M and S may be responsible for the "flavors" of the quarks. Following the current picture of the quark world if the number of colors is taken to be three then the quarks must belong to $J = 1$, the minimum value for J, and there can be a series of "generations" ($S = 0, 1, 2, \ldots$) of quarks with two distinct flavors ($M = 1, 2$). Further intrinsic differences can be induced by different possibilities for λ. Also since λ is the only distinguishing factor between different kinds of freely occurring particles it may provide the link between the confined quark world and the free lepton world. At present I am able to give only an inchoate account of how the WFDQM can add color and flavor to the quarks and keep them glued as partons of a hadron. Only further experimentation in the WFDQM laboratory, particularly with different models for λ, can show whether there is any truth in such a picture of the subnuclear spectroscopy. Anyhow I believe that the clue to the solution of the puzzles of subnuclear physics may lie in the following prophetic words of Weyl (1932), the initiator of the gauge principle, which has led to the beautiful synthesis of the theories of fundamental forces (Weinberg, 1980; Salam, 1980; Glashow, 1980).

> The kinematical structure of a physical system is expressed by an irreducible Abelian group of unitary ray rotations in system space. ...If the group is continuous this procedure automatically leads to Heisenberg's formulation. ...Our general principle allows for the possibility that the Abelian rotation group is entirely discontinuous, or that it may even be a finite group. ...But the field of discrete groups offers many possibilities which we have not yet been able to realize in Nature; perhaps these holes will be filled by applications to nuclear physics.

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